# Enumerating Arithmetical Structures on type $\mathcal{E}_n$ Graphs

Alexander Vetter

We explore the enumeration results of arithmetical structures on graphs. We present the known results of the enumeration of arithmetical structures on path graphs, cycle graphs, bidents, complete graphs, and star graphs. We then provide a new enumeration result of the number of arithmetical structures on Dynkin graphs of type E.

## 1 Introduction

This article is about the enumeration of arithmetical structures on different types of graphs. These arithmetical structures are generalizations of the Laplacian matrix of a graph and their sandpile groups. We start by recalling some definitions.

Let G be a connected graph with vertex set $V = v_1, \dots, v_n$, $D$ the diagonal matrix $\mathrm{D} = (d_i)_{i=1}^n$ where each $d_i$ is the degree of $v_i$, and $A$ the adjacency matrix of $G$. The Laplacian matrix of $G$ is $\mathrm{L} = \mathrm{D} - \mathrm{A}$, has rank $n-1$, and it is a quick exercise to check that the kernel of $D - A$ is generated by the all-ones vector $\mathbf{r} = (1)_{i=1}^n$, that is, $\mathrm{L} \cdot \mathbf{r} = \mathbf{0}$, (Section 2; 2). Thinking of the matrix $L$ as a linear transformation from $\mathbb{Z}^n \to \mathbb{Z}^n$, it has been shown that the co-kernel $\mathbb{Z}^n/\mathrm{Im}\,\mathrm{L}$ has the form $\mathbb{Z} \oplus K(G)$, where $K(G)$ is a finite abelian group called the critical group of $G$, sandpile group of $G$ or the Jacobian of $G$, (Theorem 4.2; 2).

Arithmetical structures on graphs are generalizations of the phenomenon just described. An arithmetical structure on a graph $G$ is a pair of vectors $(\mathbf{d}, \mathbf{r})$ with entries in $\mathbb{Z}_+$ such that $\mathbf{r}$ is a generator of the kernel of the matrix $L = D - A$, where $D$ is a diagonal matrix with entries given by the vector $\mathbf{d}$ and $A$ is the adjacency matrix of $G$. That is, $L \cdot \mathbf{r} = (D - A) \cdot \mathbf{r} = \mathbf{0}$. In other words, we tweak the definition of the Laplacian $L$ by allowing $\mathbf{d}$ to be any vector with positive entries and look for a generator of the kernel of $L$ with positive entries (if one exists). Note that for any arithmetical structure $(\mathbf{d}, \mathbf{r})$, the pair $(\mathbf{d}', \mathbf{r}')$, where $\mathbf{r}'$ is any positive multiple of $\mathbf{r}$, is also an arithmetical structure as $\mathbf{r}'$ would also be in the kernel of $L$. In this sense, arithmetical structures come in equivalence classes, and we usually only consider the representative $(\mathbf{d}, \mathbf{r})$ where the gcd of the entries of $\mathbf{r}$ is one.

In a seminal paper, Lorenzini studies arithmetical structures, which appeared in his research as a particular intersection of matrices when studying degenerating curves in algebraic geometry. He showed that for any graph, the number of equivalence classes of $(\mathbf{d}, \mathbf{r})$ is finite (5). Later, in 2015, a group of mathematicians working on the "Sandpile Groups" workshop in Casa Matemática in Oaxaca, MX studied and enumerated the arithmetical structures on path graphs and cycles graphs (3). Shortly after, a group of researchers of the REUF program at ICERM, studied arithmetical structures on Dynkin graphs of type $D$ (1).

The purpose of this paper is to give an accessible summary of the known results regarding the enumeration of arithmetical structures on paths, cycles, and Dynkin graphs of type $D$ and provide original results about the enumeration of arithmetical structures on a family of graphs $\mathcal{E}_n$ of type $E$. The main original result presented in this paper is the following.

**Theorem 4.2.19.** The number of arithmetical structures on $\mathcal{E}_n$ is $|Arith(\mathcal{E}_n)| = 2|Arith(\mathcal{D}_{n-1})| + |Arith(\mathcal{P}_{n-1})| - 2|Arith(\mathcal{P}_{n-2})| + \sum_{m=4}^n B(n-4, n-m)|SArith(\mathcal{E}_m)|$.

The paper is organized as follows. In Section 2, we provide background on graphs and arithmetical structures. In Section 3, we discuss the known results about the enumeration of arithmetical structures on path graphs, cycle graphs, type $D$ graphs (bidents), complete graphs, and star graphs. In Section 4, we provide new research on a family of graphs called $\mathcal{E}_n$ graphs (extensions of Dynkin graphs of type $E$) and derive a formula for the number of arithmetical structures on $\mathcal{E}_n$ in terms of the number of arithmetical structures on path graphs and bidents, as well as the number of smooth arithmetical structures on $\mathcal{E}_m$ for $4 \leq m \leq n$.

## 2 Background: Graphs and Arithmetical Structures

We begin by providing background about graphs. We then turn to our main description of arithmetical structures.

## 2.1 Graphs

We begin with some basic definitions. Let $V$ be a set of vertices and let $E$ be a set of edges between certain pairs of vertices. Then, we define a **graph** $G$ to be the pair $G = (V, E)$. We say $G$ is **connected** if there is a path between any two vertices in $G$, **undirected** if the edges of $G$ have no direction, and we say $G$ is **loop-free** if there do not exist edges from any vertex to itself. We say two vertices of $G$ are **adjacent** if they share an edge. The **degree** of a vertex $v \in V$ is the number of vertices adjacent to $v$.

For our purposes, we only consider graphs that are connected, undirected, and loop-free throughout. We now provide examples of the main graphs we consider in this paper.

**Example 2.1.1.** In Figure 1, we depict the path graph $\mathcal{P}_n$ consisting of $n$ vertices $v_1, \dots, v_n$ such that $v_i$ is adjacent to $v_{i+1}$ for $i \in \{1, \dots, n-1\}$.


*Figure 1: Path Graph $\mathcal{P}_n$ with $n$ vertices connected by the edges as indicated.*

In Figure 2, we show the graph $\mathcal{C}_n$ which consists of $n$ vertices $v_1, \dots, v_n$ such that $v_i$ is adjacent to $v_{i+1}$ for $i \in \{1,2, \dots, n\}$ where indices are taken modulo $n$.

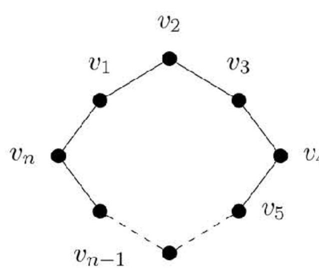



*Figure 2: Cycle Graph $\mathcal{C}_n$ with $n$ vertices connected by the edges as indicated.*

In Figure 3, we depict the graph $\mathcal{D}_n$, also denoted as a bident graph, consisting of $n = \ell + 3$ vertices $v_x, v_y, v_0, v_1, \dots, v_\ell$ such that $v_i$ is adjacent to $v_{i+1}$ for $i \in \{0,1, \dots, \ell - 1\}$, and $v_0$ is adjacent to each of $v_x$ and $v_y$.

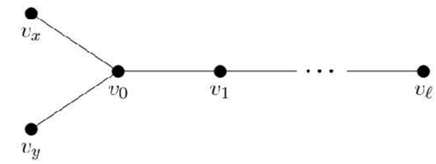



*Figure 3: Bident Graph $\mathcal{D}_n$ with $n$ vertices connected by the edges as indicated.*

In Figure 4, we depict the graph $\mathcal{E}_n$ consisting of $n = \ell + 4$ vertices $v_x, v_y, v_z, v_0, v_1, \dots, v_\ell$ such that $v_i$ is adjacent to $v_{i+1}$ for $i \in \{0,1, \dots, \ell - 1\}$, $v_0$ is adjacent to each of $v_y$ and $v_z$, and $v_x$ is adjacent to $v_z$.

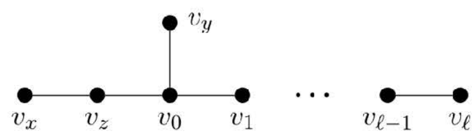



*Figure 4: Type E Graph $\mathcal{E}_n$ with $n$ vertices connected by the edges as indicated.*

## 2.2 Arithmetical Structures

We first introduce some necessary background on how to describe an arithmetical structure on a graph. Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ be any vector with $d_i \in \mathbb{R}$. Define diag($\mathbf{d}$), to be a diagonal matrix, i.e., $\operatorname{diag}(\mathbf{d}) = (e_{ij})$, where $e_{ii} = d_i$ and for $i \neq j$, $e_{ij} = 0$. For example, for $\mathbf{d} = (1,4,1,4,1,4)$,

$$\operatorname{diag}(\mathbf{d}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Let $G$ be a graph with $n$ vertices $v_1, \dots v_n$. We set $c_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and $c_{ij} = 0$ otherwise. We define the **adjacency matrix** of $G$ to be $A = (c_{ij})$. (Note that $c_{ii} = 0$ as $G$ is loop-free.) For example, the adjacency matrix $A$ for $\mathcal{C}_6$ (see Figure 2) is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We are now ready to define an arithmetical structure.

**Definition 2.2.1.** An **arithmetical structure** on a graph $G$ is a pair of positive integer vectors $(\mathbf{d}, \mathbf{r})$ along with the adjacency matrix of $G$ denoted as $A$ such that:

*1)* $\quad (diag(\mathbf{d}) - A) \cdot \boldsymbol{r} = \mathbf{0}.$

At times we will refer to $(\mathbf{d}, \mathbf{r})$ as an arithmetical $\mathbf{r}$-structure or an arithmetical $\mathbf{d}$-structure on $G$.

Note that when $\mathbf{d}$ is the vector of vertex degrees of $G$, then $\mathbf{r} = (1,1, \dots, 1)$ defines an arithmetical structure called the Laplacian arithmetical structure. Our interest is counting the number of pairs $(\mathbf{d}, \mathbf{r})$ that form an arithmetical structure on a particular graph $G$. As earlier stated, this number would be infinite because once a pair $(\mathbf{d}, \mathbf{r})$ satisfies Equation 1, then any pair $(\mathbf{d}, k \cdot \mathbf{r})$, where $k$ is a positive integer, will also satisfy the equation. Traditionally, mathematicians have imposed the condition that the entries of the $\mathbf{r}$ vector must be relatively prime, that is, there does not exist $k > 1$ such that $k | r_i$ for all $i$. An alternative approach is to define the following relation on the set of structures, $(\mathbf{d}, \mathbf{r}) \sim (\mathbf{d}', \mathbf{r}')$ if $\mathbf{d} = \mathbf{d}'$ and either $\mathbf{r} = k \cdot \mathbf{r}'$ or $\mathbf{r}' = k \cdot \mathbf{r}$ for some positive integer $k$. It is not hard to show that the transitive closure of $\sim$ is an equivalence relation. Thus, we divide the structures into equivalence classes. Each equivalence class contains a unique pair $(\mathbf{d}, \mathbf{r})$ with the entries of $\mathbf{r}$ being relatively prime. We call such $\mathbf{r}$ vectors **primitives**.

We let $Arith(G)$ be the set of all equivalence classes of arithmetical structures on $G$. Thus, we can count $|Arith(G)|$ by counting the pairs $(\mathbf{d}, \mathbf{r})$ such that $\mathbf{r}$ is primitive. A priori, it is not clear that $|Arith(G)|$ is finite, however Lorenzini proved this is the case in (5). We recall his result in Lemma 3.0.1.

Let us analyze what an arithmetical structure is at a local level. Suppose $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on a graph $G$. Then, $(\operatorname{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ implies

$$\begin{pmatrix} d_1 & -a_{1,2} & \cdots & -a_{1,n} \\ -a_{2,1} & d_2 & \cdots & -a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n,1} & -a_{n,2} & \cdots & d_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus, we can rewrite the above as a set of equations:

$$2) \qquad r_i d_i = \sum_{j \in [n] \setminus \{i\}} a_{ij}\, r_j \quad \text{for } 1 \le i \le n.$$

**Remark 2.2.2.** Using Equation 2 one can show that $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure if and only if for every vertex $v_i$, the label $r_i$ divides the sum of the $r_j$'s on adjacent vertices.

## 2.3 Examples of Arithmetical Structures

We first recall that for a graph $G$, setting $\mathbf{d}$ equal to the vector of vertex degrees in $G$ and $\mathbf{r} = (1,1, \dots, 1)$ always yield an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $G$, and $(\operatorname{diag}(\mathbf{d}) - A) = L$ is known as the Laplacian of $G$. Thus, we call $(\mathbf{d}, \mathbf{r})$ the *Laplacian arithmetical structure*. We now give three additional examples of arithmetical structures.

**Example 2.3.1.** We consider an arithmetical structure on the graph $\mathcal{P}_4$ shown in Figure 5. If $\mathbf{d} = (2,1,3,1)$ and $\mathbf{r} = (1,2,1,1)$, then $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on $\mathcal{P}_4$ since $(\operatorname{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

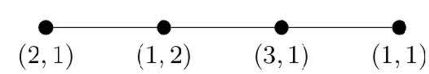



*Figure 5: An arithmetical structure on the path graph $\mathcal{P}_4$, with each vertex $v_i$ labeled by $(d_i, r_i)$.*

**Example 2.3.2.** We consider an arithmetical structure on the cycle graph $\mathcal{C}_6$ shown in Figure

6. If $\mathbf{d} = (1,4,1,4,1,4)$ and $\mathbf{r} = (2,1,2,1,2,1)$, then $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on $\mathcal{C}_6$ since $(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

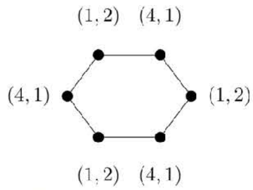



*Figure 6: An arithmetical structure on the cycle graph $\mathcal{C}_6$, with each vertex $v_i$ labeled by $(d_i, r_i)$.*

**Example 2.3.3.** We consider an arithmetical structure on the Dynkin graph $\mathcal{D}_5$ shown in Figure 7. If $\mathbf{d} = (2,1,2,3,1)$ and $\mathbf{r} = (1,2,2,1,1)$, then $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on $\mathcal{D}_5$ since $(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

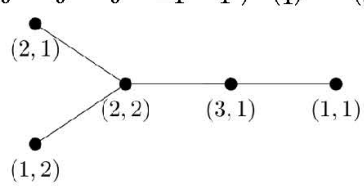



*Figure 7: An arithmetical structure on the Dynkin graph $\mathcal{D}_5$, with each vertex $v_i$ labeled by $(d_i, r_i)$.*

## 3 Previous work on Arithmetical Structures

In his seminal paper *Arithmetical Graphs* (5), Lorenzini studied arithmetical structures, which appeared in his research as a particular intersection of matrices when studying degenerating curves in algebraic geometry. The following result has motivated combinatorialists to study the enumeration of arithmetical structures.

**Lemma 3.0.1.** (Lemma 1.6; 5) There are only finitely many equivalence classes of arithmetical structures on a given graph.

The study of arithmetical structures on path graphs $\mathcal{P}_n$, cycle graphs $\mathcal{C}_n$, bident graphs $\mathcal{D}_n$, complete graphs $\mathcal{K}_n$ and star graphs $S_n$ has led to interesting combinatorial results, some of which involve the quite-famous Catalan nummmbers, binomial coefficients, and Bell numbers, as well as Egyptian fractions.

### 3.1 Results on Path Graphs

In (3), Braun et al. enumerate $Arith(G)$ when $G$ is the path graph $\mathcal{P}_n$ on $n$ vertices and when $G$ is the cycle graph $\mathcal{C}_n$ on $n$ vertices. Before going into the results, consider what an arithmetical structure looks like on a path graph in terms of matrices as described in Equation 2. Let $(\mathbf{d}, \mathbf{r})$ define an arithmetical structure on $\mathcal{P}_n$, then $(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} d_1 & -1 & 0 & 0 & 0 \\ -1 & d_2 & \ddots & 0 & 0 \\ 0 & -1 & \ddots & -1 & 0 \\ 0 & 0 & \ddots & d_{n-1} & -1 \\ 0 & 0 & 0 & -1 & d_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we can rewrite the above as a series of equations:

$$r_1 d_1 = r_2$$
$$r_i d_i = r_{i-1} + r_{i+1} \text{ for } 1 < i < n$$
$$r_n d_n = r_{n-1}.$$

Now, we begin discussing the results with the above matrix and equations in mind.

For an arithmetical $\mathbf{r}$-structure, let us define

$$\mathbf{r}(1) = \#\{i \mid r_i = 1\}.$$

Thus $\mathbf{r}(1)$ counts the number of times 1 appears in the vector $\mathbf{r}$. We now turn to one of the main results in (3).

**Theorem 3.1.1** (Theorem 3; 3)**.** The number of arithmetical structures on $\mathcal{P}_n$ is the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$. Moreover, the number of arithmetical $\mathbf{r}$-structures with $\mathbf{r}(1) = 2$ is the Catalan number $C_{n-2}$.

## 3.2 Results on Cycles Graphs

We now proceed to enumerating the arithmetical structures on cycle graphs. Before going into detail, consider what an arithmetical structure looks like on a cycle graph in terms of matrices, as discussed in Equation 2. Let $(\mathbf{d}, \mathbf{r})$ define an arithmetical structure on $\mathcal{C}_n$, then $(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} d_1 & -1 & 0 & 0 & -1 \\ -1 & d_2 & \ddots & 0 & 0 \\ 0 & -1 & \ddots & -1 & 0 \\ 0 & 0 & \ddots & d_{n-1} & -1 \\ -1 & 0 & 0 & -1 & d_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

As we can see, this is very similar to the matrix of the path graph. In fact, the only differences appear in the first and last rows of the matrix. As before, we obtain an exact formula for the number of arithmetical structures on a cycle graph. We note that $\left(\binom{n}{k}\right)$ counts all multisets of size $k$ from a set of $n$ distinct objects.

**Theorem 3.2.1.** Let $1 \leq k \leq n$ and $\ell = n - k$. Then

$$\#\{(\mathbf{d}, \mathbf{r}) \in Arith(\mathcal{C}_n) \mid \mathbf{r}(1) = k\} = \left(\binom{n}{n-k}\right) = \binom{2n-k-1}{n-k}.$$

In particular, if we sum over all $k$, we get the following total arithmetical structures on $\mathcal{C}_n$:

$$|Arith(\mathcal{C}_n)| = \sum_{k=1}^{n} \left(\binom{n}{n-k}\right) = \sum_{\ell=0}^{n-1} \left(\binom{n}{\ell}\right) = \left(\binom{n+1}{n-1}\right) = \binom{2n-1}{n-1}.$$

## 3.3 Results on Complete Graphs

The complete graph $\mathcal{K}_n$ has $n$ vertices, each pair of which is adjacent. Equivalently, $\mathcal{K}_n$ consists of $n$ vertices $v_1, \ldots, v_n$ such that there exists an edge between $v_i$ and $v_j$ for every $1 \leq i < j \leq n$. In, we depict the complete graph $\mathcal{K}_4$.

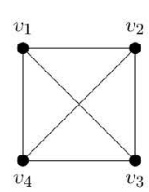



*Figure 8: Complete Graph, $\mathcal{K}_4$, with 4 vertices connected by the edges as indicated.*

Before going into the results, consider what an arithmetical structure looks like on a complete graph in terms of matrices as described in Equation 2. Let $(\mathbf{d}, \mathbf{r})$ define an arithmetical structure on $\mathcal{K}_n$, then $(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} d_1 & -1 & -1 & -1 & -1 \\ -1 & d_2 & \ddots & -1 & -1 \\ -1 & -1 & \ddots & -1 & -1 \\ -1 & -1 & \ddots & d_{n-1} & -1 \\ -1 & -1 & -1 & -1 & d_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we can rewrite the above as a set of equations:

$$r_i d_i = \sum_{j \in [n] \setminus i} r_j \text{ for } 1 \leq i \leq n.$$

Now, note the following observation:

$$\begin{aligned} 1 &= \frac{\sum_{i=1}^{n} r_i}{\sum_{i=1}^{n} r_i} \\ &= \frac{r_1}{\sum_{i=1}^{n} r_i} + \frac{r_2}{\sum_{i=1}^{n} r_i} + \cdots + \frac{r_n}{\sum_{i=1}^{n} r_i} \\ &= \frac{r_1}{r_1 + \sum_{i \in [n] \setminus 1} r_i} + \cdots + \frac{r_n}{r_n + \sum_{i \in [n] \setminus n} r_i} \\ &= \frac{r_1}{r_1 + r_1 d_1} + \frac{r_2}{r_2 + r_2 d_2} + \cdots + \frac{r_n}{r_n + r_n d_n} \\ &= \frac{r_1}{r_1(d_1 + 1)} + \frac{r_2}{r_2(d_2 + 1)} + \cdots + \frac{r_n}{r_n(d_n + 1)} \\ &= \frac{1}{d_1 + 1} + \frac{1}{d_2 + 1} + \cdots + \frac{1}{d_n + 1}. \end{aligned}$$

The last line tells us we are looking for a set of $n$ fractions whose sum is 1. These are known as Egyptian Fraction representations of 1. One can show this process is reversible and from an Egyptian Fraction we can create an arithmetical structure. In short, the number of arithmetical structures on $\mathcal{K}_n$ is the number of Egyptian Fraction representations of 1. There is no known formula for these objects; for more information about them see OEIS Sequence A002967 (4). Now, we will see that this is closely related to arithmetical structures on star graphs.

## 3.4 Results on Star Graphs

The star graph, $\mathcal{S}_n$, has $n$ vertices such that one vertex has an edge to each of the other vertices. Equivalently, $\mathcal{S}_n$ consists of vertices $v_1, \ldots, v_n$

such that there exists an edge from $v_1$ to $v_i$ for $2 \leq i \leq n$. In Figure 9, we show the graph $\mathcal{S}_5$.

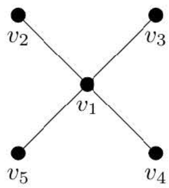



*Figure 9: Star Graph, $\mathcal{S}_5$, with 5 vertices connected by the edges as indicated.*

Before proceeding, we consider what an arithmetical structure looks like on a star graph in terms of matrices as described in Equation 2. Let $(\mathbf{d}, \mathbf{r})$ define an arithmetical structure on $\mathcal{S}_{n+1}$ with central vertex $v_0$ (why we use $n+1$ vertices will be clear shortly), then $(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} d_0 & -1 & -1 & -1 & -1 \\ -1 & d_1 & \ddots & 0 & 0 \\ -1 & 0 & \ddots & 0 & 0 \\ -1 & 0 & \ddots & d_{n-1} & 0 \\ -1 & 0 & 0 & 0 & d_n \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we can rewrite the above as a series of equations:

$$r_0 d_0 = \sum_{i=1}^{n} r_i$$
$$r_i d_i = r_0 \text{ for } 1 \leq i \leq n.$$

Now, note the following observation:

$$r_0 d_0 = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} \frac{r_0}{d_i} = r_0 \sum_{i=1}^{n} \frac{1}{d_i},$$

hence $d_0 = \sum_{i=1}^{n} \frac{1}{d_i}$. The last line tells us we are looking for a set of $n$ fractions whose sum is $d_0$. These are known as Egyptian Fraction representations of $d_0$. As with Egiptian Fractions of 1, we do not have a close formula for these numbers.

## 3.5 Results on Bidents

Archer et al. (1) examine arithmetical structures on bidents, which are paths with a fork at the end. We label the fork vertices $v_x$ and $v_y$, the vertex adjacent to both $v_x$ and $v_y$ is labeled $v_0$, and the remaining vertices are labeled in order $v_1, \ldots, v_\ell$, see Figure 10.

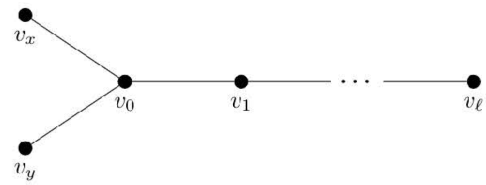



*Figure 10: Bident with labeled vertices $v_x, v_y, v_0, \cdots, v_l$.*

Before going into the results, consider what an arithmetical structure looks like on a bident in terms of matrices as done in Equation 2. Let $(\mathbf{d}, \mathbf{r})$ define an arithmetical structure on $\mathcal{D}_n$, then $(\mathrm{diag}(\mathbf{d}) - A) \cdot \mathbf{r} = \mathbf{0}$ is

$$\begin{pmatrix} d_x & 0 & -1 & 0 & 0 & 0 \\ 0 & d_y & -1 & 0 & 0 & 0 \\ -1 & -1 & d_0 & -1 & 0 & 0 \\ 0 & 0 & -1 & d_1 & \ddots & 0 \\ \vdots & \vdots & \ldots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & d_{n-3} \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_0 \\ \vdots \\ r_{n-4} \\ r_{n-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we can translate this into the following equations:

*3)*
$$\begin{aligned} d_x r_x &= r_0 \\ d_y r_y &= r_0 \\ d_0 r_0 &= r_x + r_y + r_1 \\ d_i r_i &= r_{i-1} + r_{i+1} for\ i \in \{1,2, \ldots, n-4\} \\ d_{n-3} r_{n-3} &= r_{n-4}. \end{aligned}$$

**Definition 3.5.1.** For $n \geq 4$, we call an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathcal{D}_n$ **smooth** if $d_x, d_y, d_1, d_2, \ldots, d_\ell \geq 2$. We denote by $SArith(\mathcal{D}_n)$ the set of all smooth arithmetical structures in $Arith(\mathcal{D}_n)$ where $\mathcal{D}_n$ is the bident graph with $n$ vertices.

We are now ready to present one of the main theorems in (1).

**Theorem 3.5.2** (Theorem 2.12; 1)**.** The number of arithmetical structures on $\mathcal{D}_n$ is

$$\begin{aligned} &|Arith(\mathcal{D}_n)| \\ &= 2B(n-3, n-3) \\ &+ \sum_{m=4}^{n} B(-3, -m)(|SArith(\mathcal{D}_m)| + 2), \end{aligned}$$

where $B(n, k)$ is the Ballot number

$$B(n, k) = \frac{n-k+1}{n+1} \binom{n+k}{n}.$$

Thus, the problem is reduced to counting the number of smooth arithmetical structures on

$\mathcal{D}_n$. In particular, Archer et al. show that the sequence $\{|SArith(\mathcal{D}_n)|\}_{n=4}^{\infty}$ exhibits cubic growth.

**Theorem 3.5.3.** (Theorem 4.1; 1). Let $|SArith(\mathcal{D}_n)|$ be the number of smooth arithmetical structures on $\mathcal{D}_n$. Then,

$$\frac{n^3-3n^2-n-45}{24} < |SArith(\mathcal{D}_n)| < \frac{2}{3}n^3 + \frac{4}{3}n - 8 + (2n^2-4n+2)\log(n-3).$$

Directly from (1), in Figure 11, we present a graph showing the number of smooth arithmetical structures for $n = \{4,5,\dots,50\}$, with the bounds as in Theorem 3.5.3 and the best cubic polynomial approximating the data.

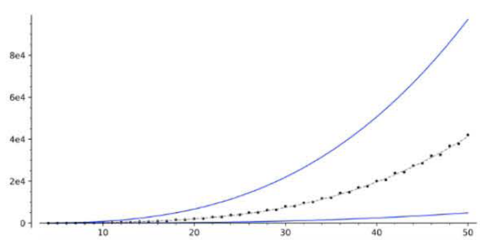



*Figure 11: The blue graphs are the upper and lower bounds of the number of smooth arithmetical structures on $\mathcal{D}_n$ provided by Theorem 3.5.3. The points $(n, |SArith(\mathcal{D}_n)|)$ are plotted with the best-fitting cubic polynomial approximating these data (Figure 6;1).*

## 4 Arithmetical Structures on type $E$ Graphs

We now consider a new set of graphs related to type $E$ Dynkin diagrams. For this reason, these graphs on $n$ vertices will be denoted $\mathcal{E}_n$.

In Figure 12, we depict the type $E$ graph $\mathcal{E}_n$ consisting of $n$ vertices $v_x, v_y, v_z, v_0 \dots, v_\ell$ where $\ell = n-4$, such that $v_i$ and $v_{i+1}$ are adjacent for $i \in \{1,\dots,\ell-1\}$, $v_0$ is adjacent to both $v_y$ and $v_z$, and $v_x$ is adjacent to $v_z$.

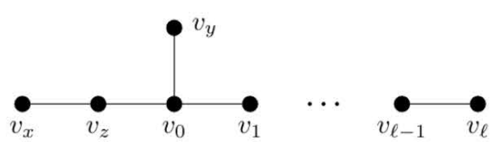



*Figure 12: Type $E$ Graph $\mathcal{E}_n$ on $n$ edges as indicated.*

We note that we allow $n = 4$, in which case we have a path on 4 vertices. We also note for $n = 5$, we have a bident on 5 vertices.

Before going into the results, consider what an arithmetical structure looks like on an $\mathcal{E}_n$ graph in terms of matrices as done in Equation 2. Let $(\mathbf{d},\mathbf{r})$ define an arithmetical structure on $\mathcal{E}_n$, $(\operatorname{diag}(\mathbf{d}) - A)\cdot\mathbf{r} = \mathbf{0}$ where $(\operatorname{diag}(\mathbf{d}) - A)\cdot\mathbf{r}$ is

$$\begin{pmatrix} d_x & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & d_z & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & d_y & -1 & 0 & \dots & 0 \\ 0 & -1 & -1 & d_0 & -1 & \vdots & 0 \\ 0 & 0 & 0 & -1 & d_1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \dots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & d_\ell \end{pmatrix}\begin{pmatrix} r_x \\ r_z \\ r_y \\ r_0 \\ r_1 \\ \vdots \\ r_\ell \end{pmatrix}$$

Now, we can translate $(\operatorname{diag}(\mathbf{d}) - A)\cdot\mathbf{r} = \mathbf{0}$ into the following equations:

*4)*
$$\begin{gathered} d_x r_x = r_z \\ d_z r_z = r_x + r_0 \\ d_0 r_0 = r_z + r_y + r_1 \\ d_y r_y = r_0 \\ d_i r_i = r_{i-1} + r_{i+1} \ \text{ for } i \in \{1,2,\dots,\ell-1\} \\ d_\ell r_\ell = r_{\ell-1}. \end{gathered}$$

### 4.1 Subdivision and Smoothing for $\mathcal{E}_n$ graphs

We now consider the operations of smoothing and *subdivision* on $\mathcal{E}_n$ analogous to the processes in (3). In fact, we will define a smoothing and subdividing operation on vertices of degree one and two in $\mathcal{E}_n$exactly in the same way Archer et al. defined smoothing and subdivision for $\mathcal{D}_n$ in (1). Let $Arith(\mathcal{E}_n)$ be the set of arithmetical structure on $\mathcal{E}_n$. Let us first consider the process of *smoothing*. Let $n \geq 4$ and let $(\mathbf{d},\mathbf{r}) \in Arith(\mathcal{E}_n)$. If $d_i = 1$ for some $i \in \{1,2,\dots,\ell-1\}$, then define vectors $\mathbf{d}'$ and $\mathbf{r}'$ of length $n-1$ as follows:

$$d_j' = \begin{cases} d_j, & \text{if } j \in \{x,y,z,0,1,\dots,i-2\} \\ d_j - 1, & \text{if } j = i-1 \\ d_{j+1} - 1, & \text{if } j = i \\ d_{j+1}, & \text{if } j \in \{i+1, i+2, \dots, \ell-1\} \end{cases}$$

and

$$r_j' = \begin{cases} r_j, & \text{if } j \in \{x,y,z,0,1,\dots,i-1\}, \\ r_{j+1}, & \text{if } j \in \{i, i+1, \dots, \ell-1\}. \end{cases}$$

As one can easily check, $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{E}_{n-1}$. We also note that we can extend this smoothing process to the degree 1 vertices in $\mathcal{E}_n$, but only if $d_j = 1$ where $j = x, y$, or $\ell$. Further, we can extend this to a smoothing of the degree 2 vertex $v_z$ if $d_z = 1$. We would decrease $d_x$ and $d_0$ by one, shift indices as we did above, and keep the $r$ values untouched other than deleting $r_z$. The smoothed structure is an arithmetical structure on a bident. For an example of the smoothing operation please refer to Example 4.1.1.

Now, we consider the process of *subdivision*. We can do this on the tail of $\mathcal{E}_n$, by which we mean the path from $v_0$ to $v_l$, including at the end of the tail. Let $n \geq 4$ and let $(\mathbf{d}, \mathbf{r}) \in Arith(\mathcal{E}_n)$. For $1 \leq i \leq \ell$, define $\mathbf{d}'$ and $\mathbf{r}'$ of length $n+1$ as follows:

$$d_j' = \begin{cases} d_j, & \text{if } j \in \{x, y, z, 0, 1, \dots, i-2\} \\ d_j + 1, & \text{if } j = i - 1 \\ 1, & \text{if } j = i \\ d_{j-1} + 1, & \text{if } j = i + 1 \\ d_{j-1}, & \text{if } j \in \{i+1, i+2, \dots, \ell + 1\} \end{cases}$$

and

$$r_j' = \begin{cases} r_j, & \text{if } j \in \{x, y, z, 0, 1, \dots, i-1\}, \\ r_{j-1} + r_j, & \text{if } j = i \\ r_{j-1}, & \text{if } j \in \{i+1, \dots, \ell+1\}. \end{cases}$$

Now, if $i = \ell + 1$, define $\mathbf{d}'$ and $\mathbf{r}'$ of length $n + 1$ as follows:

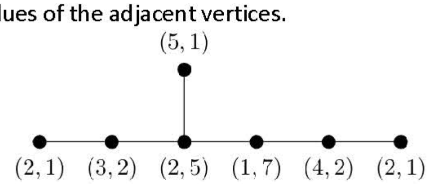
$$d_j' = \begin{cases} d_j, & \text{if } j \in \{x, y, z, 0, 1, \dots, \ell - 1\}, \\ d_j + 1, & \text{if } j = \ell \\ 1, & \text{if } j = \ell + 1 \end{cases}$$

and

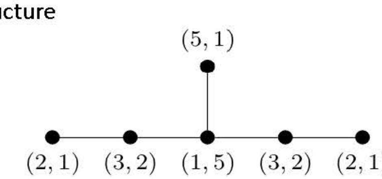
$$r_j' = \begin{cases} r_j, & \text{if } j \in \{x, y, z, 0, 1, \dots, \ell\}, \\ r_{j-1}, & \text{if } j = \ell + 1. \end{cases}$$

Then one can check that $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{E}_{n+1}$. The subdivision operation that is inverse to smoothing at $v_y$ begins with an arithmetical structure on a path graph and adds a new vertex $v_y$, connecting it to $v_0$ by a single edge and setting $r_y' = r_0$, $d_y' = 1$, and $d_0' = d_0 + 1$ while leaving the other $\mathbf{r}$-values and $\mathbf{d}$-values unchanged. We call this operation *subdivision at position* $y$. We can similarly define subdivision at position $x$. More generally, we could define a subdivision operation on an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on any graph by adding a new vertex $v_y$, connecting it by a single edge to any other vertex $v_0$ in the graph, and setting $r_y' = r_0$, $d_y' = 1$, and $d_0' = d_0 + 1$ while leaving the other $\mathbf{r}$-values and $\mathbf{d}$-values unchanged.

**Example 4.2.1.** Suppose we have the following arithmetical structure on $\mathcal{E}_7$ where we label vertex $v$ with $(d_v, r_v)$. As noted in Remark 2.2.2, each value $r_v$ must divide the sum of the $\mathbf{r}$-values of the adjacent vertices.




Since we have a vertex with $\mathbf{d}$ value 1, we can apply the smoothing operation to delete this vertex and get the following arithmetical structure




Note that the $\mathbf{d}$ values changed according to the formulas in Section 4.1. The inverse process, where we add a vertex in between two previous vertices and set the $\mathbf{r}$ value to be the sum of the $\mathbf{r}$ values of the two neighbors is the subdivision process explained in Section 4.1.

### 4.2 Results for type $\mathcal{E}_n$ Graphs

Using results from bidents as motivation, we now give results for type $\mathcal{E}_n$, graphs. We present some lemmas that will help us enumerate arithmetical structures on $\mathcal{E}_n$, in terms of arithmetical structures on paths, bidents and "partially smooth structures" on $\mathcal{E}_n$, graphs, leading to Theorem 4.2.10. This will then lead to our main theorem of the section, Theorem 4.2.19.

**Lemma 4.2.1.** The number of arithmetical structures on $\mathcal{E}_n$, with $d_x = 1$ (or equivalently $r_x = r_z$) is $|Arith(\mathcal{D}_{n-1})|$.

*Proof.* We prove this by showing there exists a bijection between structures on $\mathcal{E}_n$, graphs such that $d_x = 1$ and structures on $\mathcal{D}_{n-1}$. First, suppose we have an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathcal{E}_n$, such that $d_x = 1$. Thus, we can smooth at $v_x$. This gives us an arithmetical structure on $\mathcal{D}_{n-1}$. Now, suppose $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{D}_{n-1}$. Then, we can subdivide at $v_x$ (using the notation from Figure 10). This produces an arithmetical structure on $\mathcal{E}_n$, such that our new $d_x = 1$ (and our previous $v_x$ now becomes $v_z$). Combining, we see that these two processes are inverses of each other. This establishes our bijection. ■

**Lemma 4.2.2.** The number of arithmetical structures on $\mathcal{E}_n$, with $d_z = 1$ (or equivalently $r_z = r_x + r_0$) is $|Arith(\mathcal{D}_{n-1})|$.

*Proof.* We will show that there is a bijection between the arithmetical structures on $\mathcal{E}_n$, such that $d_z = 1$ and arithmetical structures on $\mathcal{D}_{n-1}$. First, let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathcal{E}_n$, such that $d_z = 1$. Thus, we can smooth at $v_z$ and obtain an arithmetical structure on $\mathcal{D}_{n-1}$. Now, suppose $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{D}_{n-1}$. Then, we can subdivide between $v_x$ and $v_0$ (using the notation from Figure 10) and call the new vertex $v_z$. This gives us an arithmetical structure on $\mathcal{E}_n$, such that $d_z = 1$. Combining, we see that these two processes are inverses of each other. This establishes our bijection. ■

**Lemma 4.2.3.** The number of arithmetical structures on $\mathcal{E}_n$ such that $d_y = 1$ (or equivalently $r_y = r_0$) is $Arith(|\mathcal{P}_{n-1}|)$.

*Proof.* We will show that there is a bijection between the arithmetical structures on $\mathcal{E}_n$ such that $d_y = 1$ and arithmetical structures on $\mathcal{P}_{n-1}$. First, let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathcal{E}_n$ such that $d_y = 1$. Then, we can smooth at $v_y$. Thus, we get an arithmetical structure on $\mathcal{P}_{n-1}$. Now, suppose $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{P}_{n-1}$, where we denote the vertices of $\mathcal{P}_{n-1}$ as $v_x, v_z, v_0, \cdots, v_\ell$, where $\ell = n - 4$. Then, we can subdivide at $v_0$ and call the new vertex $v_y$. This will produce an arithmetical structure on $\mathcal{E}_n$ such that $d_y = 1$. Combining, we see that these two processes are clearly inverses of each other. This establishes our bijection. ■

**Lemma 4.2.4.** There do not exist any arithmetical structures on $\mathcal{E}_n$ with $d_x = d_z = 1$.

*Proof.* In this proof, we recall the equations found in Equations 4. If $d_x = 1$, then $r_x = r_z$. But $d_z r_z = r_x + r_0$, and if $d_z = 1$, then $r_z = r_x + r_0$, or $r_z = r_z + r_0$. This implies that $r_0 = 0$, which is a contradiction since the $\mathbf{r}$ vector has strictly positive entries by definition. ■

We now note the following corollary which will become relevant later.

**Corollary 4.2.5.** There do not exist any arithmetical structures on $\mathcal{E}_n$ such that $d_x = d_y = d_z = 1$.

*Proof.* This follows from Lemma 4.2.4. ■

We now proceed to study arithmetical structures where multiple $\mathbf{d}$-values are 1.

**Lemma 4.2.6.** The number of arithmetical structures on $\mathcal{E}_n$ with $d_x = 1$ and $d_y = 1$ (or equivalently $r_x = r_z$ and $r_y = r_0$) is $Arith(|\mathcal{P}_{n-2}|)$.

*Proof.* We will show that there is a bijection between the arithmetical structures on $\mathcal{E}_n$ such that $d_x = d_y = 1$ and arithmetical structures on $\mathcal{P}_{n-2}$. First, let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathcal{E}_n$ such that $d_x = d_y = 1$. Now, we can first smooth at $v_x$. Note that this does not affect $d_y$. Now, we can smooth at $v_y$. This produces an arithmetical structure on $\mathcal{P}_{n-2}$. Now, suppose $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{P}_{n-2}$, where we denote the vertices of $\mathcal{P}_{n-2}$ as $v_z, v_0, \cdots, v_\ell$, where $\ell = n - 4$. Then, we can subdivide at vertex $v_z$ and call the new vertex $v_x$ to get an arithmetical structure on $\mathcal{P}_{n-1}$ such that $d_x = 1$. Now, subdivide at vertex $v_0$ and call the new vertex $v_y$ to get an arithmetical structure on $\mathcal{E}_n$ such that $d_x = d_y = 1$. Combining, we see that these two processes are clearly inverses of each other. This establishes our bijection. ■

**Lemma 4.2.7.** The number of arithmetical structures on $\mathcal{E}_n$ such that $d_z = 1$ and $d_y = 1$ (or equivalently $r_z = r_x + r_0$ and $r_y = r_0$) is $|Arith(\mathcal{P}_{n-2})|$.

*Proof.* We will show that there is a bijection between the arithmetical structures on $\mathcal{E}_n$ such that $d_z = d_y = 1$ and arithmetical structures on $\mathcal{P}_{n-2}$. First, let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathcal{E}_n$ such that $d_z = d_y = 1$. Now, we can first smooth at $v_z$. Note that this does not affect $d_y$. Now, we can smooth at $v_y$. This produces an arithmetical structure on $\mathcal{P}_{n-2}$. Now, suppose $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{P}_{n-2}$, where we denote the vertices of $\mathcal{P}_{n-2}$ as $v_x, v_0, \cdots, v_\ell$, where $\ell = n - 4$. Then, we can subdivide between $v_x$ and $v_0$and call the new vertex $v_z$ to get an arithmetical structure on $\mathcal{P}_{n-1}$such that $d_z = 1$. Now, subdivide at vertex $v_0$ and call the new vertex $v_y$ to get an arithmetical structure on $\mathcal{E}_n$ such that $d_z = d_y = 1$. Combining, we see that these two processes are clearly inverses of each other. This establishes our bijection. ■

The previous lemmas give us a way to characterize the arithmetical structures on $\mathcal{E}_n$ such that at least one of $d_x$, $d_y$, and $d_z$ is equal to 1. The following definition characterizes all remaining possibilities.

**Definition 4.2.8.** We call an arithmetical structure on $\mathcal{E}_n$ **partially smooth** if none of $d_x, d_y$, and $d_z$ equals 1. We denote the set of such partially smooth structures as $PSArith(\mathcal{E}_n)$.

**Definition 4.2.9.** We call an arithmetical structure on $\mathcal{E}_n$ **smooth** if $d_i \geq 2$ for all $i \geq 1$. We denote the set of such smooth structures as $SArith(\mathcal{E}_n)$).

We will later show that the conditions in Definition 4.2.9 imply $r_0 > r_1 > \cdots > r_\ell$ (see Lemma 4.2.13). We now enumerate arithmetical structures of $\mathcal{E}_n$ in terms of bidents, paths, and partially smooth structures on $\mathcal{E}_n$.

**Theorem 4.2.10**. The number of arithmetical structures on $\mathcal{E}_n$ is the following:

$$\begin{aligned} |Arith(\mathcal{E}_n)| = 2|Arith(\mathcal{D}_{n-1})| \\ + |Arith(\mathcal{P}_{n-1})| \\ - 2|Arith(\mathcal{P}_{n-2})| \\ + |PSArith(\mathcal{E}_n)|. \end{aligned}$$

*Proof.* We use the lemmas we developed in this section to prove this result. First, consider arithmetical structures such that $d_x = 1$. By Lemma 4.2.1, the number of such arithmetical structures is $|Arith(\mathcal{D}_{n-1})|$. Next, consider arithmetical structures such that $d_z = 1$. By Lemma 4.2.2, the number of such arithmetical structures is $|Arith(\mathcal{D}_{n-1})|$. Next, consider arithmetical structures such that $d_y = 1$. By Lemma 4.2.3, the number of such arithmetical structures is $|Arith(\mathcal{P}_{n-1})|$. Now, these sets do not necessarily have an empty intersection. By Lemma 4.2.4, the intersection of the first two is the empty set. Further, by Corollary 4.2.5, the intersection of all 3 is the empty set. By Lemma 4.2.6, the number of arithmetical structures such that $d_x = d_y = 1$ is $|Arith(\mathcal{P}_{n-2})|$. By Lemma 4.2.7, the number of arithmetical structures such that $d_z = d_y = 1$ is $|Arith(\mathcal{P}_{n-2})|$. Thus, the total number of arithmetical structures such that at least one of $d_x$, $d_y$, and $d_z$ equals 1 is equal to: $2|Arith(\mathcal{D}_{n-1})| + |Arith(\mathcal{P}_{n-1})| - 2|Arith(\mathcal{P}_{n-2})|$.

Now, note that these arithmetical structures on $\mathcal{E}_n$ we just described and arithmetical structures on $PSArith(\mathcal{E}_n)$ have an empty intersection since, by definition, the values of $d_x, d_y$, and $d_z$ are greater than one in partially smooth structures. Further, all arithmetical structures on $\mathcal{E}_n$ fall into one of those two categories. Thus,

$$\begin{aligned}|Arith(\mathcal{E}_n)| = 2|Arith(\mathcal{D}_{n-1})| \\ + |Arith(\mathcal{P}_{n-1})| \\ - 2|Arith(\mathcal{P}_{n-2})| \\ + |PSArith(\mathcal{E}_n)|. \blacksquare\end{aligned}$$

We now work our way toward counting partially smooth structures.

**Definition 4.2.11.** If an arithmetical structure $(\mathbf{d}', \mathbf{r}')$ on $\mathcal{E}_m$ (for $m < n$) can be obtained from an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathcal{E}_n$ by a sequence of smoothing operations, then we say $(\mathbf{d}', \mathbf{r}')$ is an **ancestor** of $(\mathbf{d}, \mathbf{r})$. If an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathcal{E}_n$ can be obtained from an arithmetical structure $(\mathbf{d}', \mathbf{r}')$ on $\mathcal{E}_m$ (for $m < n$) by a sequence of subdivision operations, then we say $(\mathbf{d}', \mathbf{r}')$ is a **descendant** of $(\mathbf{d}, \mathbf{r})$.

In particular, $(\mathbf{d}', \mathbf{r}')$ is a descendant of $(\mathbf{d}, \mathbf{r})$ if and only if $(\mathbf{d}, \mathbf{r})$ is an ancestor of $(\mathbf{d}', \mathbf{r}')$ .

**Lemma 4.2.12.** Every partially smooth arithmetical structure on $\mathcal{E}_n$ is either smooth or has a unique smooth ancestor of one of the following graphs:

a) An $\mathcal{E}_m$ graph for some $m \geq 5$,
b) A path on four vertices (which we also denote as $\mathcal{E}_4$).

*Proof.* Let $(\mathbf{d}, \mathbf{r})$ be a partially smooth arithmetical structure on $\mathcal{E}_n$. In particular, this means $d_x, d_y$, and $d_z$ are greater than 1. If $(\mathbf{d}, \mathbf{r})$ is not smooth, then $d_i = 1$ for some $i \in \{1, \dots, \ell\}$. Then, perform a smoothing operation at $v_i$. Repeat this process until we have an arithmetical structure, $(\mathbf{d}', \mathbf{r}')$ on $\mathcal{E}_{n-s}$ where $s$ is the number of times we perform a smoothing operation, such that $d_i' > 1$ for all $i \geq 1$. During each smoothing, we eliminate a vertex $v_i$ such that $r_{i-1}, r_{i+1} < r_i$ (since $d_i = 1$). This means our remaining $\mathbf{r}' = (r_0', r_1', \dots, r_{\ell-s}')$ is the maximal decreasing subsequence of $r_0, r_1, \dots, r_\ell$ because the entries of $\mathbf{r}$ do not change aside from the deleted elements. Hence, the vector $\mathbf{r}'$ is unique.

Once this process terminates, only $d_0$ may equal 1, which we allow. Thus, at this point, we cannot perform a smoothing operation. We note that we end with an $\mathcal{E}_m$ graph, where $m \geq 4$, as desired. ■

We now generalize Lemma 2.1 in (1) to $\mathcal{E}_n$ graphs.

**Lemma 4.2.13.** Let $n \geq 5$ and $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on $\mathcal{E}_n$. The following are equivalent:

a) $d_i \geq 2$ for all $i \in \{1,2,3, \dots, \ell\}$.
b) $r_0 > r_1 > \cdots > r_{\ell-1} > r_\ell$.
c) $r_0 - r_1 \geq r_1 - r_2 \geq \cdots \geq r_{l-2} - r_{\ell-1} \geq r_{\ell-1} - r_\ell > 0$.

*Proof.* We first show that a) implies c). Using our matrix description from Equation 3, we see that if $i \in \{1, \dots, \ell - 1\}$, then $d_i r_i = r_{i-1} + r_{i+1}$, which implies $d_i r_i - r_i = r_{i-1} + r_{i+1} - r_i$. Thus, $(d_i - 1)r_i - r_{i+1} = r_{i-1} - r_i$, but since $d_i \geq 2$, it follows that $r_{i-1} - r_i \geq r_i - r_{i+1}$. Thus, we see that

$r_0 - r_1 \geq r_1 - r_2 \geq \cdots \geq r_{\ell-2} - r_{\ell-1} \geq r_{\ell-1} - r_\ell$.

We also know that $r_{\ell-1} = d_\ell r_\ell \geq 2r_\ell$, hence $r_{\ell-1} - r_\ell > 0$.

Now we show that c) implies b). Since $r_{i-1} - r_i > 0$ for all $i \in \{1, \dots, \ell\}$, we see that

$r_0 > r_1 > \cdots > r_{\ell-1} > r_\ell$.

Now we show that b) implies a). Let $i \in \{1, \dots, \ell - 1\}$, and since $r_i < r_{i-1}$, we must have $r_i < r_{i-1} + r_{i+1} = d_i r_i$. But we know $d_i$ is an integer, so $d_i \geq 2$ for all $i \in \{1, \dots, \ell - 1\}$. Further, $r_\ell < r_{\ell-1} = d_\ell r_\ell$, hence, $d_\ell \geq 2$. ■

We now establish results analogous to those found for $\mathcal{D}_n$ in Section 2.3 of (1). Let $(\mathbf{d}^0, \mathbf{r}^0)$ be an arithmetical structure on $\mathcal{E}_m$ with $m \leq n$.

**Definition 4.2.14.** For $n \geq 4$, a sequence of positive integers $\mathbf{b} = (b_1, \dots, b_{n-m})$ is a **valid**

**subdivision sequence** for $(\mathbf{d}^0, \mathbf{r}^0)$ if its entries satisfy $1 \le b_i \le m - 4 + i$.

We can inductively define an arithmetical structure $Sub\big((\mathbf{d}^0, \mathbf{r}^0), b\big)$ on $\mathcal{E}_n$. Let $\big(\mathbf{d}^i, \mathbf{r}^i\big)$ be the arithmetical structure on $\mathcal{E}_{m+i}$ after subdividing at $v_{b_i-1}$ on the arithmetical structure $\big(\mathbf{d}^{i-1}, \mathbf{r}^{i-1}\big)$ on $\mathcal{E}_{m+i-1}$. We note that this is possible because $1 \le b_i \le m - 4 + i$, so on the $i^{th}$ step, the tail length is $m - 4 - i$, and we are always allowed to subdivide from position 1 (vertex $v_0$)to the end of the tail (vertex $v_{m-4}$). Now, let

$$Sub\big((\mathbf{d}^0, \mathbf{r}^0), \mathbf{b}\big) := (\mathbf{d}^{n-m}, \mathbf{r}^{n-m}).$$

Further, the descendants of $(\mathbf{d}^0, \mathbf{r}^0)$ are exactly $Sub\big((\mathbf{d}^0, \mathbf{r}^0), \mathbf{b}\big)$ for some $\mathbf{b}$.

**Lemma 4.2.15.** Let $4 \le m \le n$, let $(\mathbf{d}^0, \mathbf{r}^0)$ be an arithmetical structure on $\mathcal{E}_m$, and let $\mathbf{b} = (b_1, \dots, b_{n-m})$ be a valid subdivision sequence for $(\mathbf{d}^0, \mathbf{r}^0)$. Suppose $j$ is a positive integer satisfying $1 \le j < n - m$ with $b_j > b_{j+1}$. Define $\mathbf{b}' = (b_1', b_2', \dots, b_{n-m}')$ by

$$b_i' = \begin{cases} b_{j+1}, & \text{if } i = j, \\ b_j + 1, & \text{if } i = j + 1 \\ b_i, & \text{otherwise.} \end{cases}$$

Then, $Sub\Big(\big(\mathbf{d}^0, \mathbf{r}^0\big), \mathbf{b}\Big) = Sub\big((\mathbf{d}^0, \mathbf{r}^0), \mathbf{b}'\big)$.

*Proof.* Note that this is analogous to Lemma 2.7 in (1) since subdivision only occurs on the tail of the type $E$ structure. ∎

Note that this lemma implies the order in which we subdivide along the tail does not matter except for when the subdivisions are adjacent to one another. The next result is an analogue of Proposition 14 in (3) and Lemma 2.8 in (1). It states that we may identify each descendant of an arithmetical structure $(\mathbf{d}^0, \mathbf{r}^0)$ by a unique valid non-decreasing subdivision sequence.

**Lemma 4.2.16.** Let $(\mathbf{d}^0, \mathbf{r}^0)$ be an arithmetical structure on $\mathcal{E}_m$with $d_i^0 \ge 2$ for every $i \in \{1,2, \dots, m-4\}$. Then there is a bijection between arithmetical structures on $\mathcal{E}_n$ that are descendants of $\big(\mathbf{d}^0, \mathbf{r}^0\big)$ and valid subdivision sequences $\mathbf{b} = (b_1, b_2, \dots, b_{n-m})$ that satisfy $b_i \le b_{i+1}$ for all $i$.

*Proof.* Suppose $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on $\mathcal{E}_n$ that is a descendant of $(\mathbf{d}^0, \mathbf{r}^0)$. This implies $(\mathbf{d}, \mathbf{r}) = Sub\big((\mathbf{d}^0, \mathbf{r}^0), \mathbf{b}'\big)$ for some $\mathbf{b}' = (b_1', \dots, b_{n-m}')$ such that $1 \le b_i' \le m - 4 + i$ for all $i$. Through repeated applications of Lemma 4.2.15, we get $(\mathbf{d}, \mathbf{r}) = Sub\big((\mathbf{d}^0, \mathbf{r}^0), \mathbf{b}\big)$ for some sequence $\mathbf{b}$ that satisfies the claim.

We note that at each stage of subdivision, $b_i$ is the largest value of $j$ such that $d_j^i = 1$. Thus, if we start with $(\mathbf{d}, \mathbf{r})$ and subdivide at position $j$, where $j$ is the largest number such that $d_j = 1$, we recover $\mathbf{b}$. This implies we have a unique sequence for each descendant of $(\mathbf{d}^0, \mathbf{r}^0)$. ∎

We can use the above result to count arithmetical structures. Before this, we need one more result. For this purpose, let $B(n, k)$ denote the so-called **ballot numbers**, i.e.,

$$B(n, k) = \frac{n - k + 1}{n + 1}\binom{n + k}{n}.$$

These are a generalization of the Catalan numbers. For more details on ballot numbers, see (6). The next result appeared as Lemma 2.10 in (1).

**Lemma 4.2.17** (Lemma 2.10; 1)**.** Fix $4 \le m \le n$. There are $B(n - 4, n - m)$ valid subdivision sequences $\mathbf{b} = (b_1, b_2, \dots, b_{n-m})$ such that $b_i \le b_{i+1}$ for all $i$.

**Proposition 4.2.18.** Let $n \ge 4$ and let $des_n(G; \mathbf{d}, \mathbf{r})$ be the number of arithmetical structures on $\mathcal{E}_n$ that are descendants of the arithmetical structure $(\mathbf{d}, \mathbf{r})$ on a given subgraph $G$ of $\mathcal{E}_n$. Then

$$des_n(\mathcal{E}_m; \mathbf{d}, \mathbf{r}) = B(n - 4, n - m)$$

for every smooth arithmetical structure $(\mathbf{d}, \mathbf{r})$ on $\mathcal{E}_n$ for any $4 \le m \le n$.

*Proof.* We appeal to Lemma 4.2.16 which gives a bijection between arithmetical structures on

$\mathcal{E}_n$ that are descendants of arithmetical structures on $\mathcal{E}_m$ and sequences $(b_1, \ldots, b_{n-m})$ such that $1 \le b_i \le m - 4 + i$ and $b_i \le b_{i+1}$ for all i. Then, Lemma 4.2.17 applies and we get a total of $B(n-4, n-m)$ such smooth arithmetical structures. ■

Since every partially smooth arithmetical structure has a unique smooth ancestor, we can classify each of them by their subdivision sequences. These subdivision sequences are thus able to help us count our total structures. In particular, each smooth structure on a given $\mathcal{E}_m$ produces the same number of partially smooth structures for some $n \ge m$. Using the previous results, we can count the number of arithmetical structures on $\mathcal{E}_n$ in terms of smooth arithmetical structures on $\mathcal{E}_m$. We are now ready to prove our main theorem.

**Theorem 4.2.19.** The number of arithmetical structures on $\mathcal{E}_n$ is

$$|Arith(\mathcal{E}_n)| = 2|Arith(\mathcal{D}_{n-1})| + |Arith(\mathcal{P}_{n-1})| - 2|Arith(\mathcal{P}_{n-2})| + \sum_{m=4}^{n} B(n-4, n-m)|SArith(\mathcal{E}_m)|.$$

*Proof.* In light of Theorem 4.2.10, we only need to focus on counting partially smooth structures. Proposition 4.2.18 gives $B(n-4, n-m)$ descendant arithmetical structures for the smooth arithmetical structures on $\mathcal{E}_m$. Thus, we obtain a total of

$$\sum_{m=4}^{n} B(n-4, n-m)|SArith(\mathcal{E}_m)|$$

from these subgraphs. Substituting into Theorem 4.2.10, we obtain the desired result. ■

The sets $Arith(\mathcal{P}_{n-1})$, $Arith(\mathcal{P}_{n-2})$, and $Arith(\mathcal{D}_{n-1})$ have been studied in Section 3. Thus, at this point, the problem is reduced to counting the number of smooth arithmetical structures on $\mathcal{E}_n$ for all $n \ge 4$.

### 4.3 A Step Toward Enumerating Smooth Arithmetical Structures on $\mathcal{E}_n$

Now we focus on smooth structures, which we divide into two categories: those where $d_0 = 1$ and those where $d_0 \ne 1$. Theorem 4.3.2 gives a complete characterization of these structures.

**Lemma 4.3.1.** Let $(\boldsymbol{d}, \boldsymbol{r})$ be a smooth arithmetical structure on $\mathcal{E}_m$ with $m \ge 5$. Then $d_0 = 1$ if and only if $r_0 > r_y + r_z$. Moreover, for a structure $(\boldsymbol{d}, \boldsymbol{r})$ on $\mathcal{E}_4$, $d_0 = 1$ if and only if $r_0 = r_y + r_z$.

*Proof.* Suppose first that $d_0 = 1$, then $r_0 = r_z + r_y + r_1 > r_y + r_z$. Conversely, suppose $r_0 > r_y + r_z$. By the definition of an arithmetical structure, $r_0$ must divide $r_y + r_z + r_1$, so $r_1 = -(r_y + r_z) \bmod r_0$. Hence, $r_1 = pr_0 - (r_y + r_z)$ for some positive integer $p$. Since $(\mathbf{d}, \mathbf{r})$ is smooth, then by Lemma 4.2.13, $r_0 > r_1 = pr_0 - (r_y + r_z)$. This, together with the assumption that $r_0 > r_y + r_z$, implies $p = 1$ and $r_0 = r_1 + r_y + r_z$. Therefore, $d_0 = 1$. ■

The last sentence of the lemma is a restatement of the definition of an arithmetical structure on a graph $\mathcal{E}_4$.

We now derive an equation whose parameters we will investigate using Equations 4. Let $(\mathbf{d}, \mathbf{r})$ be a smooth arithmetical structure (so none of the entries of the $\mathbf{d}$ vector can be one except for $d_0$) on $\mathcal{E}_n$. This means $r_z = kr_x$ for some $k \ge 2$. Recall that $d_z \ge 2$, so let $d_z = q + 1$, $q \ge 1$. Thus,

$$r_x + r_0 = d_z r_z = (q+1)kr_x,$$

hence

$$r_0 = (k - 1 + qk)r_x. \tag{5}$$

We also know that $r_y | r_0$, $r_0 = d_y r_y > r_y$, and $gcd(r_x, r_y) = 1$ (otherwise our $\mathbf{r}$-vector would not be primitive). With these conditions, $r_y$ is a factor of $(k - 1 + qk)$, and this factor must be relatively prime to $r_x$.

**Theorem 4.2.3.** Let $n \ge 5$ and let $(\mathbf{d}, \mathbf{r})$ be a smooth arithmetical structure on $\mathcal{E}_n$. Then, $d_0 = 1$ except for the following family of exceptional cases:

$\mathbf{r} = (2, 2k, 4k - 2, 2k - 1, 4k - 3, 4k - 4, \cdots, 2, 1)$

and

$$\mathbf{d} = (k, 2, 2, 2, 2, 2, \cdots, 2, 2),$$

on $\mathcal{E}_{4k+1}$ for any $k \ge 2$.

*Proof.* By Lemma 4.3.1, if $(\mathbf{d},\mathbf{r})$ is a structure on $\mathcal{E}_m$, for $m \geq 5$, we have $d_0 = 1$ if and only if $r_0 > kr_x + r_y = r_z + r_y$.
We split the proof in three cases. We will make use of Equation 5.

**Case 1** ($r_x = 1$): Note that $r_z = kr_x = k$ and $r_0 = k - 1 + qk$. Since $d_y r_y = r_0$ and $d_y > 1$, then

$$r_y \leq \frac{k-1+qk}{2} = \frac{r_0}{2}.$$

Using this, we now claim that if $1 < k(q-1)$ then $r_0 > r_z + r_y$ which would imply that $d_0 = 1$. To prove the claim, note that

$$\begin{aligned} 1 < k(q-1) &\Leftrightarrow k < qk - 1 \\ &\Leftrightarrow k - 1 + qk < 2qk - 2 \\ &\Leftrightarrow \frac{k-1+qk}{2} < qk - 1 \\ &\Leftrightarrow k + \frac{k-1+qk}{2} < k + qk - 1 \\ &\Rightarrow r_z + r_y < r_0. \end{aligned}$$

If $q \geq 2$ and $k \geq 2$ then $1 < k(q-1)$, so $r_z + r_y < r_0$. If $q = 1$ and $k \geq 2$, then $r_0 = 2k - 1$. Since $d_y r_y = r_0 = 2k - 1$ and $d_y > 1$ then $r_y \leq \frac{2k-1}{3}$. In this case,

$$\begin{aligned} r_0 = 2k - 1 &\geq k + 2/3k + 1/3k - 1 \\ &\geq k + \frac{2k-1}{3} \geq r_z + r_y, \end{aligned}$$

with equality only when $k = 2$. By investigating this case, we find it produces a structure on $\mathcal{E}_4$. This structure is excluded from the theorem as $d_0 = 1$ for this structure. In particular, $r_x = 1$, $q = 1$, and $k = 2$, which gives us the following structure on $\mathcal{E}_4$: $\mathbf{r} = (1,2,3,1)$ and $\mathbf{d} = (2,2,1,3)$. This finishes Case 1.
For the next two cases, since $r_0 = \big((k-1) + qk\big)r_x$, $d_y r_y = r_0$ and $gcd(r_x, r_y) = 1$ then the largest possible value of $r_y$ is $(k-1) + qk$. Note that if $(qk-1)(r_x - 1) \geq k$ were to hold then

$$\begin{aligned} (qk-1)(r_x-1) &\geq k \\ \Leftrightarrow (qk-1)r_x &\geq k - 1 + qk \\ \Leftrightarrow (k-1)r_x + qkr_x &\geq kr_x + (k-1) + qk \\ \Rightarrow r_0 &\geq r_z + r_y. \end{aligned} \tag{6}$$

We will use this to investigate the cases when $r_x \geq 2$.

**Case 2** ($r_x = 2$): Since $k \geq 2$, if $q \geq 2$, then $(qk-1)(2-1) > k$ so $r_0 > r_z + r_y$ by Equation 6. If $q = 1$ and $r_y \leq \big((k-1) + qk\big)/2$ then

$$\begin{aligned} r_0 &= \big((k-1) + k\big)2 \\ &= 4k - 2 \\ &> 2k + \big((k-1) + k\big)/2 \\ &\geq r_z + r_y. \end{aligned}$$

Finally, if $q = 1$ and $r_y$ is the largest possible value it could attain, i.e. $r_y = (k-1) + qk = 2k - 1$, then $r_0 = 2(2k-1)$ and $r_z + r_y = 2k + (k-1) + k$, hence $r_0 < r_z + r_y$. In this case, we get a smooth arithmetical structure with $d_0 > 1$ for each $k \geq 2$. Particularly, we get the structure with
$\mathbf{r} = (2, 2k, 4k-2, 2k-1, 4k-3, 4k-4, \cdots, 2, 1)$
and

$$\mathbf{d} = (k, 2,2,2,2,2, \cdots, 2,2),$$

as shown in Figure 13, which has length $4k+1$.

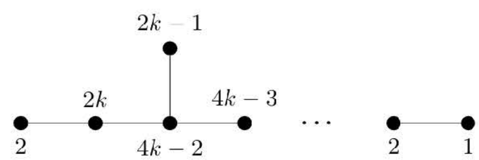



*Figure 13: A smooth arithmetical structure on $\mathcal{E}_{4k+1}$ with $d_0 = 2$. We label each vertex v simply by $r_v$.*

**Case 3** ($r_x \geq 3$): In this case,

$$(qk-1)(r_x-1) \geq (qk-1)(2) \geq k,$$

so by Equation 6, $r_0 \geq r_z + r_y$. Note that equality only occurs when $r_x = 3, q = 1, k = 2$. In this case, we get the $\mathbf{r}$ vector $\mathbf{r} = (3,6,9,3)$, which is in the same equivalence class as the arithmetical structure of $\mathcal{E}_4$ found in Case 1. Otherwise, $(qk-1)(r_x-1) > k$ so $r_0 > r_z + r_y$. Hence, $d_0 = 1$. ■

## 5 Future Work

### 5.1 Enumeration of Smooth Arithmetical Structures on $\mathcal{E}_n$

Theorem 4.3.2 will be key in the enumeration of smooth arithmetical structures on $\mathcal{E}_n$. Particularly, since $d_0 = 1$ in all but the one exceptional family, we have that in all other smooth structures $(\mathbf{d},\mathbf{r})$, $r_1 = r_0 - r_z - r_y$.

Once $r_x, r_y, r_z$, and $r_0$, have been determined (or equivalently, once parameters $r_x, r_y, k \geq 2$ (so $r_z = kr_x$), and $q \geq 1$ (so $r_0 = (k-1) + qkr_x$) have been chosen), a slight alteration of Proposition 2.4 in (1) shows that there is a unique structure with these values and this structure has length $n = 3 + F(r_0, r_1)$, where $F$ is the function defined in (Section 3.1; 1).

To investigate this value $F(r_0, r_1)$, it will be helpful to consider the following rescaling of the $\mathbf{r}$ vector. We recall that arithmetical structures come in equivalence classes and $\mathbf{r}$ vectors that are integer multiples of each other belong to the same class. Thus, to make things simpler, we are going to manipulate our $\mathbf{r}$-vector.

Suppose $(\mathbf{d}, \mathbf{r})$ is a smooth arithmetical structure on $\mathcal{E}_n$ with $d_0 = 1$. Note that $r_z = kr_x$, $r_x | r_0$, $r_y | r_z$, and $gcd(r_x, r_y) = 1$. Thus, $c = \frac{r_0}{r_x r_y}$ is an integer. Set a new vector $r' = c \cdot r$. This gives

$$\begin{aligned} r_x' &= cr_x = \frac{r_0}{r_y} =: a \\ r_z' &= cr_z = k\frac{r_0}{r_y} = ka \\ r_y' &= cr_y = \frac{r_0}{r_x} =: b \\ r_0' &= cr_0 = \frac{r_0 r_0}{r_x r_y} = ab. \end{aligned}$$

Since $r_1 = r_0 - r_y - r_z$, we get that

$$\begin{aligned} r_1' = cr_1 = c(r_0 - r_y - r_z) &= r_0' - r_y' - r_z' \\ &= ab - b - ka. \end{aligned}$$

The fact that $r_1' = cr_1 > 0$ implies that $k < b$, a bound that is useful for computational purposes. Moreover, using Lemma 3.2 in (1), we get

$$\begin{aligned} F(r_0, r_1) = F(cr_0, cr_1) &= F(r_0', r_1') \\ &= F(ab, ab - b - ka) \\ &= F\big(ab, ab - (ka + b)\big). \end{aligned}$$

The key insight in (1) to provide an algorithm to count all smooth arithmetical structures on $D_n$ was an analysis to the expression $F(ab, ab - a - b)$ done in (Theorem 3.3; 1). Our expression here, has an extra parameter $k$ that satisfies $2 \leq k < b$. It would be interesting to work on a generalization of the arguments in (1) to find a similar algorithm that will count the number of smooth arithmetical structures on $\mathcal{E}_n$ with $d_0 = 1$. This, together with the one exceptional family in Theorem 4.2.3 would provide a full count of all smooth arithmetical structures on $\mathcal{E}_n$. Theorem 4.2.19 would then provide a full count of all arithmetical structures on $\mathcal{E}_n$.


### Author

**Alexander Vetter**

Alexander Vetter is a Mathematics PhD student at the University of Pennsylvania. He was a Presidential Scholar, Mathematics major, and Honors major at Villanova University where he graduated in 2019. During his time at Villanova, he worked on several research projects and participated in the Budapest Semesters in Mathematics and Math in Moscow study abroad programs. His current research is in Algebraic Combinatorics.



### Mentor

**Alexander Diaz-Lopez**

Alexander Diaz-Lopez is an award-winning teacher and math researcher currently working at Villanova University. Born and raised in the archipelago of Puerto Rico, Alexander attended the University of Puerto Rico, Mayaguez and then moved to the University of Notre Dame to obtain his PhD in Mathematics. After working at Swarthmore College for one year, he joined the Department of Mathematics and Statistics at Villanova University as a tenure-track assistant professor in 2017.



**ACKNOWLEDGEMENTS**

The author would like to thank his thesis mentor, Dr. Alexander Diaz-Lopez, for his continual guidance and support throughout this project and Dr. Andrew Woldar for reviewing his Honor Thesis from which this work was developed and providing invaluable feedback. Finally, the authors thanks the Villanova University Honors Department, Dr. Allison Payne, and Madeline Reynolds for coordinating the Honors Thesis Program.


**REFERENCES**


1. Archer, K., Bishop, A. C., Diaz-Lopez, A., García Puente, L. D., Glass, D., Louwsma, J. Arithmetical structures on bidents. *Discrete Math.* 343 (2020), no. 7, 111850, 23 pp.
2. Biggs, N. L. Chip-firing and the critical group of a graph. *J. Algebraic Combin.* 9 (1999), no. 1, 25–45.
3. Braun, B, Corrales, H., Corry, S., García Puente, L. D., Glass, D., Kaplan, N., Martin, J. L., Musiker, G., Valencia, C. E. Counting arithmetical structures on paths and cycles. *Discrete Math.* 341 (2018), no. 10, 2949–2963.
4. OEIS Foundation Inc. *The On-line Encyclopedia of Integer Sequences*. http://oeis.org/A002967 (2019).
5. Lorenzini, D. J. Arithmetical graphs. *Math. Ann.* 285 (1989), no. 3, 481–501.
6. Stanley, R. P. *Catalan numbers*. Cambridge University Press, New York, 2015. viii+215 pp. ISBN: 978-1-107-42774-7.